

## GRANIČNE VREDNOSTI FUNKCIJA

### zadaci II deo

U sledećim zadacima ćemo koristiti poznatu graničnu vrednost:

$$\boxed{\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1}$$
 ali i manje “varijacije”

$$\lim_{x \rightarrow 0} \frac{\sin ax}{ax} = 1 \quad \text{i} \quad \lim_{x \rightarrow 0} \frac{\sin^n ax}{(ax)^n} = 1$$

### Zadaci:

1) Odrediti sledeće granične vrednosti:

a)  $\lim_{x \rightarrow 0} \frac{\sin 4x}{x}$ ;

b)  $\lim_{x \rightarrow 0} \frac{\operatorname{tg} x}{x}$ ;

v)  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$ ;

g)  $\lim_{x \rightarrow a} \frac{\sin x - \sin a}{x - a}$ ;

### Rešenja:

a)  $\lim_{x \rightarrow 0} \frac{\sin 4x}{x}$ ; (i gore i dole dodamo 4)  $= \lim_{x \rightarrow 0} \frac{4 \sin 4x}{4x} = 4 \cdot \boxed{\lim_{x \rightarrow 0} \frac{\sin 4x}{4x}} = 4 \cdot 1 = 4$

Ovde smo “napravili” i upotreбили da je  $\lim_{x \rightarrow 0} \frac{\sin ax}{ax} = 1$

b)  $\lim_{x \rightarrow 0} \frac{\operatorname{tg} x}{x} = \lim_{x \rightarrow 0} \frac{\frac{\sin x}{\cos x}}{x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} = \lim_{x \rightarrow 0} \boxed{\frac{\sin x}{x}} \cdot \frac{1}{\cos x} = \lim_{x \rightarrow 0} 1 \cdot \frac{1}{\cos x}$   
 $= 1 \cdot \lim_{x \rightarrow 0} \frac{1}{\cos x} = 1 \cdot \frac{1}{\cos 0} = 1 \cdot \frac{1}{1} = 1$

v)  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} =$  iskoristićemo formulu iz trigonometrije:  $1 - \cos x = 2 \sin^2 \frac{x}{2}$   
 $= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x^2} =$  (i gore i dole dodamo 4)  $= \lim_{x \rightarrow 0} 2 \cdot \frac{4 \sin^2 \frac{x}{2}}{x^2} = 8 \cdot \boxed{\lim_{x \rightarrow 0} \frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2}} = 8 \cdot 1 = 8$

$$g) \lim_{x \rightarrow a} \frac{\sin x - \sin a}{x - a} = \text{iskoristićemo formulu ( pogledaj PDF fajl iz II godine )}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$= \lim_{x \rightarrow a} \frac{2 \cos \frac{x+a}{2} \sin \frac{x-a}{2}}{x-a} = \text{malo prisredimo...}$$

$$= \lim_{x \rightarrow a} \cos \frac{x+a}{2} \cdot \boxed{\frac{\sin \frac{x-a}{2}}{\frac{x-a}{2}}} = \lim_{x \rightarrow a} \cos \frac{x+a}{2} \cdot 1 =$$

$$= \cos \frac{a+a}{2} = \cos \frac{2a}{2} = \cos a$$

## 2) Izračunati sledeće granične vrednosti:

a)  $\lim_{x \rightarrow 0} \frac{\sin 4x}{\sqrt{x+1}-1}$ ;

b)  $\lim_{x \rightarrow \pi} \frac{\cos \frac{x}{2}}{x-\pi}$ ;

v)  $\lim_{x \rightarrow 1} \frac{\sin(1-x)}{\sqrt{x}-1}$ ;

a)

$$\lim_{x \rightarrow 0} \frac{\sin 4x}{\sqrt{x+1}-1} = \text{najpre racionalizacija}$$

$$= \lim_{x \rightarrow 0} \frac{\sin 4x}{\sqrt{x+1}-1} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} =$$

$$= \lim_{x \rightarrow 0} \frac{\sin 4x \cdot (\sqrt{x+1}+1)}{x+1-1} = \lim_{x \rightarrow 0} \frac{\sin 4x \cdot (\sqrt{x+1}+1)}{x}$$

sad i gore i dole dodamo 4

$$= \lim_{x \rightarrow 0} \frac{4 \sin 4x (\sqrt{x+1}+1)}{4x} = \lim_{x \rightarrow 0} 4 \boxed{\frac{\sin 4x}{4x}} (\sqrt{x+1}+1) = \lim_{x \rightarrow 0} 4 \cdot 1 \cdot (\sqrt{x+1}+1) =$$

$$= 4(\sqrt{0+1}+1) = 4 \cdot 2 = 8$$

b)

$$\lim_{x \rightarrow \pi} \frac{\cos \frac{x}{2}}{x - \pi} = \text{ovde } \acute{\text{c}}\text{emo najpre uzeti smenu: } x - \pi = t, \text{ pa kad } x \rightarrow \pi, \text{ onda } t \rightarrow \pi - \pi = 0, \text{ dakle } \boxed{t \rightarrow 0}$$

$$\lim_{t \rightarrow 0} \frac{\cos \frac{t + \pi}{2}}{t} = \lim_{t \rightarrow 0} \frac{\cos\left(\frac{\pi}{2} + \frac{t}{2}\right)}{t} = \lim_{t \rightarrow 0} \frac{-\sin \frac{t}{2}}{t} \quad (\text{jer je } \cos\left(\frac{\pi}{2} + \alpha\right) = -\sin \alpha)$$

$$= -\lim_{t \rightarrow 0} \frac{\sin \frac{t}{2}}{2 \cdot \frac{t}{2}} = -\lim_{t \rightarrow 0} \frac{1}{2} \cdot \frac{\sin \frac{t}{2}}{\frac{t}{2}} = -\frac{1}{2} \cdot 1 = -\frac{1}{2}$$

v)

$$\lim_{x \rightarrow 1} \frac{\sin(1-x)}{\sqrt{x}-1} = \text{najpre racionalizacija}$$

$$\lim_{x \rightarrow 1} \frac{\sin(1-x)}{\sqrt{x}-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} = \lim_{x \rightarrow 1} \frac{\sin(1-x)(\sqrt{x}+1)}{x-1} = \text{sada smena } x-1=t, \text{ kad } x \rightarrow 1 \text{ tad } t \rightarrow 0$$

$$= \lim_{t \rightarrow 0} \frac{\sin(-t)(\sqrt{t+1}+1)}{t} = \lim_{t \rightarrow 0} \frac{-\sin(t)(\sqrt{t+1}+1)}{t} = -\lim_{t \rightarrow 0} \frac{\sin t}{t} (\sqrt{t+1}+1)$$

$$= -\lim_{t \rightarrow 0} 1 \cdot (\sqrt{t+1}+1) = -(1+1) = -2$$

**U sledećim zadacima ćemo koristiti:**

$$\boxed{\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e} \quad \text{I} \quad \lim_{x \rightarrow \infty} \left(1 + \frac{1}{ax}\right)^{ax} = e$$

Još nam treba i činjenica da je  $e^x$  neprekidna funkcija i važi:

$$\boxed{\lim_{x \rightarrow a} e^{f(x)} = e^{\lim_{x \rightarrow a} f(x)}}$$

**3) Odrediti sledeće granične vrednosti:**

a)  $\lim_{x \rightarrow \infty} \left(1 + \frac{3}{x}\right)^x$ ;

b)  $\lim_{x \rightarrow \infty} \left(\frac{x+1}{x-1}\right)^x$ ;

c)  $\lim_{x \rightarrow \infty} x(\ln(x+1) - \ln x)$ ;

**Rešenja:**

a)

$\lim_{x \rightarrow \infty} \left(1 + \frac{3}{x}\right)^x$  = ovde gde je 3 mora biti 1, pa ćemo 3 'spustiti' ispod x

$\lim_{x \rightarrow \infty} \left(1 + \frac{3}{x}\right)^x = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{\frac{x}{3}}\right)^x$  = sad kod x u eksponentu pomnožimo i podelimo sa 3

$$\lim_{x \rightarrow \infty} \left(1 + \frac{3}{x}\right)^x = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{\frac{x}{3}}\right)^x = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{\frac{x}{3}}\right)^{\frac{x}{3} \cdot 3} = \boxed{\lim_{x \rightarrow \infty} \left(1 + \frac{1}{\frac{x}{3}}\right)^{\frac{x}{3}}} = e^3$$

b)

$\lim_{x \rightarrow \infty} \left(\frac{x+1}{x-1}\right)^x$  = trik: u zagradi ćemo dodati 1 i oduzeti 1 =

$$\lim_{x \rightarrow \infty} \left(1 + \frac{x+1}{x-1} - 1\right)^x = \lim_{x \rightarrow \infty} \left(1 + \frac{x+1-1(x-1)}{x-1}\right)^x = \lim_{x \rightarrow \infty} \left(1 + \frac{x+1-x+1}{x-1}\right)^x$$

$$= \lim_{x \rightarrow \infty} \left(1 + \frac{2}{x-1}\right)^x = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{\frac{x-1}{2}}\right)^x = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2} \cdot \frac{2}{x-1} \cdot x} =$$

$$= \lim_{x \rightarrow \infty} \left(1 + \frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2} \cdot \frac{2x}{x-1}} = \boxed{\lim_{x \rightarrow \infty} \left(1 + \frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2} \cdot \frac{2x}{x-1}}} = \lim_{x \rightarrow \infty} e^{\frac{2x}{x-1}} = e^{\lim_{x \rightarrow \infty} \frac{2x}{x-1}} = e^2$$

v)

$$\lim_{x \rightarrow \infty} x \cdot (\ln(x+1) - \ln x) = \lim_{x \rightarrow \infty} \left[x \cdot \ln \frac{x+1}{x}\right] = \lim_{x \rightarrow \infty} \ln \left(\frac{x+1}{x}\right)^x =$$

( pošto je ln neprekidna funkcija i ona može da zameni mesto sa lim )

$$\ln \lim_{x \rightarrow \infty} \left(\frac{x+1}{x}\right)^x = \ln \lim_{x \rightarrow \infty} \left(\frac{x}{x} + \frac{1}{x}\right)^x = \ln \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = \ln e = 1$$

Ovde smo koristili pravila (pogledaj II godina logaritmi):  $\ln A - \ln B = \ln \frac{A}{B}$  i  $n \cdot \ln A = \ln A^n$

4) Odrediti sledeće granične vrednosti:

a)  $\lim_{x \rightarrow 0} (1 + 3 \operatorname{ctg}^2 x)^{\operatorname{ctg}^2 x} = ?$

b)  $\lim_{x \rightarrow 0} (\cos x)^{\frac{1}{\sin^2 x}} = ?$

**Rešenja:**

a)  $\lim_{x \rightarrow 0} (1 + 3 \operatorname{ctg}^2 x)^{\operatorname{ctg}^2 x} = ?$

$$\lim_{x \rightarrow 0} (1 + 3 \operatorname{ctg}^2 x)^{\operatorname{ctg}^2 x} = \lim_{x \rightarrow 0} \left(1 + 3 \cdot \frac{1}{\operatorname{ctg}^2 x}\right)^{\operatorname{ctg}^2 x} = \lim_{x \rightarrow 0} \left(1 + \frac{1}{\frac{\operatorname{ctg}^2 x}{3}}\right)^{\operatorname{ctg}^2 x} =$$

$$\lim_{x \rightarrow 0} \left(1 + \frac{1}{\frac{\operatorname{ctg}^2 x}{3}}\right)^{\operatorname{ctg}^2 x \cdot \frac{3}{3}} = \lim_{x \rightarrow 0} \left(1 + \frac{1}{\frac{\operatorname{ctg}^2 x}{3}}\right)^{\frac{\operatorname{ctg}^2 x}{3} \cdot 3} = \boxed{\lim_{x \rightarrow 0} \left(1 + \frac{1}{\frac{\operatorname{ctg}^2 x}{3}}\right)^{\frac{\operatorname{ctg}^2 x}{3}}} = e^3$$

b)  $\lim_{x \rightarrow 0} (\cos x)^{\frac{1}{\sin^2 x}} = ?$

Najpre ćemo dodati i oduzeti jedinicu...

$$\lim_{x \rightarrow 0} (\cos x)^{\frac{1}{\sin^2 x}} = \lim_{x \rightarrow 0} (1 + \cos x - 1)^{\frac{1}{\sin^2 x}}$$

Dalje moramo upotrebiti formulu:  $1 - \cos x = 2 \sin^2 \frac{x}{2}$

$$\begin{aligned} \lim_{x \rightarrow 0} (\cos x)^{\frac{1}{\sin^2 x}} &= \lim_{x \rightarrow 0} (1 + \cos x - 1)^{\frac{1}{\sin^2 x}} = \lim_{x \rightarrow 0} (1 - (1 - \cos x))^{\frac{1}{\sin^2 x}} = \lim_{x \rightarrow 0} \left(1 - 2 \sin^2 \frac{x}{2}\right)^{\frac{1}{\sin^2 x}} = \lim_{x \rightarrow 0} \left(1 - \frac{1}{2 \sin^2 \frac{x}{2}}\right)^{\frac{1}{\sin^2 x}} = \\ &= \lim_{x \rightarrow 0} \left(1 + \frac{1}{\frac{1}{2 \sin^2 \frac{x}{2}}}\right)^{\frac{1}{\sin^2 x}} = \{ \text{formula } \sin^2 x = 4 \sin^2 \frac{x}{2} \cos^2 \frac{x}{2} \} = \lim_{x \rightarrow 0} \left(1 + \frac{1}{-\frac{1}{4 \sin^2 \frac{x}{2} \cos^2 \frac{x}{2}}}\right)^{\frac{1}{\sin^2 x}} = \lim_{x \rightarrow 0} \left(1 + \frac{1}{-\frac{1}{\sin^2 \frac{x}{2} 4 \cos^2 \frac{x}{2}}}\right)^{\frac{1}{\sin^2 x}} = \\ &= \lim_{x \rightarrow 0} \left(1 + \frac{1}{-\frac{1}{2 \cos^2 \frac{x}{2}}}\right)^{\frac{1}{\sin^2 \frac{x}{2} 2}} = \lim_{x \rightarrow 0} \frac{-1}{2 \cos^2 \frac{x}{2}} = e^{-\frac{1}{2}} \end{aligned}$$

Ko je upoznat sa Lopitalovom teoremom može ove zadačice rešavati i na drugi način:

a)  $\lim_{x \rightarrow 0} (1 + 3 \operatorname{tg}^2 x)^{\operatorname{ctg}^2 x} = ?$

Ceo limes obeležimo sa nekim slovom, recimo A i **elenujemo** ga:

$$\lim_{x \rightarrow 0} (1 + 3 \operatorname{tg}^2 x)^{\operatorname{ctg}^2 x} = A \dots \dots \dots / \ln$$

$$\ln \lim_{x \rightarrow 0} (1 + 3 \operatorname{tg}^2 x)^{\operatorname{ctg}^2 x} = \ln A$$

$$\lim_{x \rightarrow 0} \ln (1 + 3 \operatorname{tg}^2 x)^{\operatorname{ctg}^2 x} = \ln A$$

$$\lim_{x \rightarrow 0} \operatorname{ctg}^2 x \cdot \ln (1 + 3 \operatorname{tg}^2 x) = \ln A$$

$$\lim_{x \rightarrow 0} \frac{1}{\operatorname{tg}^2 x} \ln (1 + 3 \operatorname{tg}^2 x) = \ln A$$

$$\lim_{x \rightarrow 0} \frac{\ln (1 + 3 \operatorname{tg}^2 x)}{\operatorname{tg}^2 x} = \ln A \quad \text{sad na levoj strani upotrebljavamo Lopitalovu teoremu}$$

$$\lim_{x \rightarrow 0} \frac{\frac{1}{1 + 3 \operatorname{tg}^2 x} \cdot 3 \cdot 2 \operatorname{tg} x \cdot \frac{1}{\cos^2 x}}{2 \operatorname{tg} x \cdot \frac{1}{\cos^2 x}} = \ln A$$

$$\lim_{x \rightarrow 0} \frac{3}{1 + 3 \operatorname{tg}^2 x} = \ln A \rightarrow \frac{3}{1 + 3 \operatorname{tg}^2 0} = \ln A \rightarrow \frac{3}{1} = \ln A \rightarrow \ln A = 3 \rightarrow \boxed{A = e^3}$$

b)  $\lim_{x \rightarrow 0} (\cos x)^{\frac{1}{\sin^2 x}} = ?$

$$\lim_{x \rightarrow 0} (\cos x)^{\frac{1}{\sin^2 x}} = A \dots \dots \dots / \ln$$

$$\ln \lim_{x \rightarrow 0} (\cos x)^{\frac{1}{\sin^2 x}} = \ln A$$

$$\lim_{x \rightarrow 0} \ln (\cos x)^{\frac{1}{\sin^2 x}} = \ln A$$

$$\lim_{x \rightarrow 0} \frac{1}{\sin^2 x} \ln (\cos x) = \ln A$$

$$\lim_{x \rightarrow 0} \frac{\ln (\cos x)}{\sin^2 x} = \ln A \quad \text{na levoj strani Lopital...}$$

$$\lim_{x \rightarrow 0} \frac{\frac{1}{\cancel{\cos x}} (-\cancel{\sin x})}{2 \cancel{\sin x} \cos x} = \ln A$$

$$\lim_{x \rightarrow 0} \frac{-1}{2 \cos^2 x} = \ln A \rightarrow \frac{-1}{2 \cos^2 0} = \ln A \rightarrow \frac{-1}{2 \cdot 1} = \ln A \rightarrow \frac{-1}{2} = \ln A \rightarrow \boxed{A = e^{-\frac{1}{2}}}$$

**Vi naravno radite kako zahteva vaš profesor...**

Kao što vidite, Lopitalova teorema je **elegantan** način da se dodje do rešenja kod odredjivanja graničnih vrednosti funkcija. Ali pazite, ona radi samo u situacijama  $\frac{0}{0}$  i  $\frac{\infty}{\infty}$ .

**5) Odrediti sledeće granične vrednosti:**

a)  $\lim_{x \rightarrow 0} x^2 \ln x$

b)  $\lim_{x \rightarrow 0} x \cdot \text{ctg } 2x$

**Rešenja:**

a)  $\lim_{x \rightarrow 0} x^2 \ln x$

Ako zamenimo da  $x$  teži nuli, dobijamo :  $\lim_{x \rightarrow 0} x^2 \ln x = 0^2 \cdot \ln 0 = 0 \cdot (-\infty)$

Ovo je neodređen izraz a **ne smemo** koristiti Lopitalovu teoremu . **Šta uraditi?**

Moramo prepraviti funkciju od koje tražimo limes da bude oblika  $\frac{0}{0}$  ili  $\frac{\infty}{\infty}$ .

$$\lim_{x \rightarrow 0} x^2 \ln x = \lim_{x \rightarrow 0} \frac{\ln x}{\frac{1}{x^2}} = \text{ako ovde zamenimo da } x \text{ teži nuli, dobijamo:}$$

$$\lim_{x \rightarrow 0} x^2 \ln x = \lim_{x \rightarrow 0} \frac{\ln x}{\frac{1}{x^2}} = \frac{\ln 0}{\frac{1}{0^2}} = \frac{-\infty}{\infty}, \text{ pa možemo koristiti Lopitala...}$$

$$\lim_{x \rightarrow 0} x^2 \ln x = \lim_{x \rightarrow 0} \frac{\ln x}{\frac{1}{x^2}} = \lim_{x \rightarrow 0} \frac{(\ln x)'}{\left(\frac{1}{x^2}\right)'} = \lim_{x \rightarrow 0} \frac{\frac{1}{x}}{\frac{-2}{x^3}} = \lim_{x \rightarrow 0} \frac{x^{\cancel{2}}}{-2 \cancel{x}} = \lim_{x \rightarrow 0} \frac{x^2}{-2} = 0$$

b)  $\lim_{x \rightarrow 0} x \cdot \text{ctg } 2x$

Sličan trik kao u prethodnom primeru...

$$\lim_{x \rightarrow 0} x \cdot \text{ctg } 2x = \lim_{x \rightarrow 0} \frac{\text{ctg } 2x}{\frac{1}{x}} = \left(\frac{\infty}{\infty}\right) = \lim_{x \rightarrow 0} \frac{(\text{ctg } 2x)'}{\left(\frac{1}{x}\right)'} = \lim_{x \rightarrow 0} \frac{-\frac{1}{\sin^2 2x} \cdot 2}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0} \frac{2x^2}{\sin^2 2x} = \left(\frac{0}{0}\right)$$

Opet koristimo Lopitalovu teoremu...

$$\lim_{x \rightarrow 0} \frac{2x^2}{\sin^2 2x} = \left(\frac{0}{0}\right) = \lim_{x \rightarrow 0} \frac{(2x^2)'}{(\sin^2 2x)'} = \lim_{x \rightarrow 0} \frac{4x}{2 \sin 2x \cdot \cos 2x \cdot 2} = \lim_{x \rightarrow 0} \frac{\cancel{4}x}{\cancel{4} \sin 2x \cdot \cos 2x} = \lim_{x \rightarrow 0} \frac{x}{\sin 2x \cdot \cos 2x} = \left(\frac{0}{0}\right)$$

Auuu, opet Lopital...

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x}{\sin 2x \cdot \cos 2x} &= \left(\frac{0}{0}\right) = \lim_{x \rightarrow 0} \frac{1}{[(\sin 2x)' \cdot \cos 2x + \sin 2x \cdot (\cos 2x)']} = \\ \lim_{x \rightarrow 0} \frac{1}{[\cos 2x \cdot 2 \cdot \cos 2x + \sin 2x \cdot (-\sin 2x) \cdot 2]} &= \lim_{x \rightarrow 0} \frac{1}{2 \cdot \cos^2 2x - 2 \sin^2 2x} = \frac{1}{2 \cdot \cos^2 2 \cdot 0 - 2 \sin^2 2 \cdot 0} = \\ &= \frac{1}{2 \cdot \cos^2 0 - 2 \sin^2 0} = \frac{1}{2 \cdot 1 - 0} = \frac{1}{2} \end{aligned}$$

Ovaj zadatak baš ispade težak, zar ne?

**Al to je zato što ne razmišljamo, već odmah krenemo u rad...**

Evo kako bi moglo prostije:

$$\lim_{x \rightarrow 0} x \cdot \operatorname{ctg} 2x = \lim_{x \rightarrow 0} x \cdot \frac{1}{\operatorname{tg} 2x} = \lim_{x \rightarrow 0} \frac{x}{\operatorname{tg} 2x} = \left(\frac{0}{0}\right) = \lim_{x \rightarrow 0} \frac{1}{\frac{1}{\cos^2 2x} \cdot 2} = \lim_{x \rightarrow 0} \frac{\cos^2 2x}{2} = \frac{\cos^2 0}{2} = \frac{1}{2}$$

**Dakle, prvo pogledajte malo zadatak, analizirajte, pa onda krenite na rešavanje...**