

TRIGONOMETRIJSKE FUNKCIJE POLUUGLOVA

Formule su:

$$1. \sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}} \quad \text{ili} \quad 2 \sin^2 \frac{\alpha}{2} = 1 - \cos \alpha$$

$$2. \cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}} \quad \text{ili} \quad 2 \cos^2 \frac{\alpha}{2} = 1 + \cos \alpha$$

$$3. \operatorname{tg} \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$$

$$4. \operatorname{ctg} \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{1 - \cos \alpha}}$$

Primeri:

1) Odrediti $\cos \frac{\alpha}{2}$, ako je $\sin \alpha = \frac{4}{5}$ i $\alpha \in \left(-\frac{3\pi}{2}, -\pi\right)$

Pošto je $\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$ moramo naći $\cos \alpha$.

$$\sin^2 \alpha + \cos^2 \alpha = 1$$

$$\cos^2 \alpha = 1 - \sin^2 \alpha$$

$$\cos^2 \alpha = 1 - \left(\frac{4}{5}\right)^2$$

$$\cos^2 \alpha = 1 - \frac{16}{25}$$

$$\cos^2 \alpha = \frac{9}{25}$$

$$\cos^2 \alpha = \pm \sqrt{\frac{9}{25}}$$

$$\cos \alpha = \pm \frac{3}{5}$$

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

$$\cos \frac{\alpha}{2} = -\sqrt{\frac{1 - \frac{3}{5}}{2}}$$

$$\cos \frac{\alpha}{2} = -\sqrt{\frac{\frac{2}{5}}{2}}$$

$$\cos \frac{\alpha}{2} = -\frac{1}{\sqrt{5}} \quad \text{racionališemo}$$

$$\cos \frac{\alpha}{2} = -\frac{\sqrt{5}}{5}$$

Pošto je $\alpha \in \left(-\frac{3\pi}{2}, -\pi\right)$

iz $\alpha \in \left(-\frac{3\pi}{2}, -\pi\right) \Rightarrow \frac{\alpha}{2} \in \left(-\frac{3\pi}{4}, -\frac{\pi}{2}\right)$

To nam govori da je iz II kvadranta

$$\cos \alpha = -\frac{3}{5}$$

2) Odrediti $\cos \frac{\alpha}{2}$, ako je $\sin \alpha = -\frac{4\sqrt{2}}{9}$ i $\alpha \in \left(\pi, \frac{3\pi}{2}\right)$

Moramo najpre naći $\cos \alpha$

$$\cos^2 \alpha = 1 - \sin^2 \alpha$$

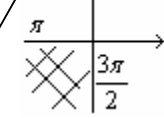
$$\cos^2 \alpha = 1 - \left(-\frac{4\sqrt{2}}{9}\right)^2$$

$$\cos^2 \alpha = 1 - \frac{16 \cdot 2}{81}$$

$$\cos^2 \alpha = 1 - \frac{32}{81}$$

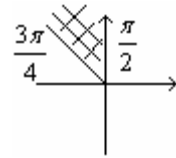
$$\cos^2 \alpha = \frac{49}{81}$$

$$\cos \alpha = \pm \frac{7}{9}$$

$$\alpha \in \left(\pi, \frac{3\pi}{2}\right) \Rightarrow$$


$$\cos \alpha = -\frac{7}{9}$$

$$\alpha \in \left(\pi, \frac{3\pi}{2}\right) \rightarrow \frac{\alpha}{2} \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$$



Za $\sin \frac{\alpha}{2}$ uzimamo +

$$\sin \frac{\alpha}{2} = +\sqrt{\frac{1 - \cos \alpha}{2}} = +\sqrt{\frac{1 + \frac{7}{9}}{2}}$$

$$\sin \frac{\alpha}{2} = +\frac{4}{3\sqrt{2}}$$

$$\sin \frac{\alpha}{2} = \frac{4}{3\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{3 \cdot \cancel{2}}$$

$$\sin \frac{\alpha}{2} = \frac{2\sqrt{2}}{3}$$

3) Dokazati:

a) $tg \frac{\alpha}{2} = \frac{1 - \cos \alpha}{\sin \alpha}$ za $\alpha \neq \pi(2k+1), k \in \mathbb{Z}$

b) $tg \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha}$ za $\alpha \neq \pi(2k+1), k \in \mathbb{Z}$

a) Podjimo sad od desne strane da dokažemo levu. $\left[\sin \alpha = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \right]$

$$\frac{1 - \cos \alpha}{\sin \alpha} = \frac{\cancel{2} \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{\cancel{2} \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} = \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} = tg \frac{\alpha}{2}$$

b)
$$\frac{\sin \alpha}{1 + \cos \alpha} = \frac{\cancel{2} \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{\cancel{2} \cos \frac{\alpha}{2} \cancel{1} \frac{\alpha}{2}} = \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} = tg \frac{\alpha}{2}$$

Odakle $2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} = \sin \alpha$?

Pa iz: $\sin 2\alpha = 2 \sin \alpha \cos \alpha$

4) Izračunati vrednost izraza $A = \frac{\sin x + 2 \cos x}{\operatorname{tg} x - \operatorname{ctg} x}$, ako je $\operatorname{tg} \frac{x}{2} = 2$

Iskoristićemo $\operatorname{tg} \frac{x}{2} = 2$, da nađemo $\cos x$

$$\operatorname{tg} \frac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}}$$

$$\sqrt{\frac{1 - \cos x}{1 + \cos x}} = 2 \rightarrow \text{Kvadriramo}$$

$$\frac{1 - \cos x}{1 + \cos x} = 4$$

$$1 - \cos x = 4(1 + \cos)$$

$$1 - \cos x = 4 + 4 \cos x$$

$$-\cos x - 4 \cos x = 4 - 1$$

$$-5 \cos x = 3$$

$$\cos x = -\frac{3}{5}$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\sin^2 x = 1 - \frac{9}{25}$$

$$\sin^2 x = \frac{16}{25}$$

$$\sin x = \pm \frac{4}{5}$$

$$\sin x = +\frac{4}{5}$$

$$\operatorname{tg} x = \frac{\sin x}{\cos x} = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$$

$$\operatorname{ctg} x = -\frac{3}{4}$$

Izračunajmo vrednost izraza A

$$A = \frac{\sin x + 2 \cos x}{\operatorname{tg} x - \operatorname{ctg} x}$$

$$A = \frac{\frac{4}{5} + 2 \cdot \left(-\frac{3}{5}\right)}{-\frac{4}{3} + \frac{3}{4}}$$

$$A = \frac{-\frac{2}{5}}{-\frac{7}{12}}$$

$$A = \frac{24}{35}$$

5) Dokaži indetitete:

$$a) \frac{1 + \sin x - \cos x}{1 + \sin x + \cos x} = \operatorname{tg} \frac{x}{2}$$

$$b) \frac{2 \sin x - \sin 2x}{2 \sin x + \sin 2x} = \operatorname{tg}^2 \frac{x}{2}$$

Rešenje: Naravno, podjemo od leve strane pa transformišemo izraz dok ne dodjemo do desne strane!!

$$a) \frac{1 + \sin x - \cos x}{1 + \sin x + \cos x} = \frac{1 - \cos x + \sin x}{1 + \cos x + \sin x} \quad \text{Ideja je: } \begin{cases} 1 - \cos x = 2 \sin^2 \frac{x}{2} \\ 1 + \cos x = 2 \cos^2 \frac{x}{2} \\ \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} \end{cases}$$

$$= \frac{2 \sin^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}} = \text{Izvučemo zajednički gore i dole!!}$$

$$= \frac{\cancel{2} \sin \frac{x}{2} \left(\cancel{\sin \frac{x}{2}} + \cancel{\cos \frac{x}{2}} \right)}{\cancel{2} \cos \frac{x}{2} \left(\cancel{\cos \frac{x}{2}} + \cancel{\sin \frac{x}{2}} \right)}$$

$$= \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \operatorname{tg} \frac{x}{2}$$

b)

$$\frac{2 \sin x - \sin 2x}{2 \sin x + \sin 2x} = \frac{2 \sin x - 2 \sin x \cos x}{2 \sin x + 2 \sin x \cos x} =$$

$$= \frac{\cancel{2 \sin x} (1 - \cos x)}{\cancel{2 \sin x} (1 + \cos x)} =$$

$$= \frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} = \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} = \operatorname{tg}^2 \frac{x}{2}$$

6) Izračunati bez upotrebe računskih pomagala $tg 7^{\circ}30'$

Ideja nam je da iskoristimo malopre dokazanu jednakost: $tg \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha}$

$$tg 7^{\circ}30' = tg \frac{15^{\circ}}{2} = \frac{\sin 15^{\circ}}{1 + \cos 15^{\circ}}$$

Sada moramo naći $\sin 15^{\circ}$ i $\cos 15^{\circ}$

$$\sin 15^{\circ} = \sin \frac{30^{\circ}}{2} = \sqrt{\frac{1 - \cos 30^{\circ}}{2}} = \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2 - \sqrt{3}}{4}}$$

$$\sin 15^{\circ} = \frac{\sqrt{2 - \sqrt{3}}}{2}, \text{ ovo je tačno, al je malo komplikovano zbog duplog korena}$$

koji bi morali da ‘uništimo’ preko Lagranžovog indetiteta (pogledaj to), zato ćemo ići:

$$\begin{aligned} \sin 15^{\circ} &= \sin(45^{\circ} - 30^{\circ}) = \sin 45^{\circ} \cos 30^{\circ} - \cos 45^{\circ} \sin 30^{\circ} = \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{2}(\sqrt{3} - 1)}{4} \end{aligned}$$

$$\begin{aligned} \cos 15^{\circ} &= \cos(45^{\circ} - 30^{\circ}) = \cos 45^{\circ} \cos 30^{\circ} + \sin 45^{\circ} \sin 30^{\circ} = \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{2}(\sqrt{3} + 1)}{4} \end{aligned}$$

Sad je:

$$tg 7^{\circ}30' = \frac{\frac{\sqrt{2}(\sqrt{3} - 1)}{4}}{1 + \frac{\sqrt{2}(\sqrt{3} + 1)}{4}}$$

$$tg 7^{\circ}30' = \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} + \sqrt{2} + 5} = \text{odradimo duplu racionalizaciju (pogledaj to u delu}$$

korenovanje)

$$\boxed{tg 7^{\circ}30' = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2}$$

7) Ako je $tg \frac{x}{2} = t$ izračunati $\sin x$, $\cos x$ i tgx “preko” t ,

Rešenja: (ovo će nam biti smena kod trigonometrijskih integrala, zato obratiti pažnju!!!)

$$\begin{aligned} \sin x &= \frac{\sin x}{1} = \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}} = \frac{\cancel{\cos^2 \frac{x}{2}} \left(\frac{\sin \frac{x}{2}}{2 \frac{\cos \frac{x}{2}}{2}} \right)}{\cancel{\cos^2 \frac{x}{2}} \left(\frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} + 1 \right)} \\ &= \frac{2tg \frac{x}{2}}{tg^2 \frac{x}{2} + 1} = \frac{2t}{t^2 + 1} = \frac{2t}{1 + t^2} \end{aligned}$$

$$\begin{aligned} \cos x &= \frac{\cos x}{1} = \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}} = \frac{\cancel{\cos^2 \frac{x}{2}} \left(\frac{\sin^2 \frac{x}{2}}{1 - \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}}} \right)}{\cancel{\cos^2 \frac{x}{2}} \left(\frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} + 1 \right)} \\ &= \frac{1 - tg^2 \frac{x}{2}}{1 + tg^2 \frac{x}{2}} = \frac{1 - t^2}{1 + t^2} \end{aligned}$$

$$tgx = \frac{\sin x}{\cos x} = \frac{\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}} = \frac{2t}{1-t^2}$$

8) Izračunati vrednost izraza $\frac{\sin 160^\circ}{\cos^4 40^\circ - \sin^4 40^\circ} =$

$$\frac{\sin 160^\circ}{\cos^4 40^\circ - \sin^4 40^\circ} = (\text{ovo dole je razlika kvadrata})$$

$$\frac{\sin 160^\circ}{(\cos^2 40^\circ + \sin^2 40^\circ)(\cos^2 40^\circ - \sin^2 40^\circ)} = \text{ovde je } \cos^2 40^\circ + \sin^2 40^\circ = 1$$

$$\frac{\sin 160^\circ}{\cos^2 40^\circ - \sin^2 40^\circ} = (\text{ovo dole je } \cos^2 x - \sin^2 x = \cos 2x)$$

$$\frac{\sin 160^\circ}{\cos 80^\circ} = \frac{2 \sin 80^\circ \cancel{\cos 80^\circ}}{\cancel{\cos 80^\circ}} = 2 \sin 80^\circ$$

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